Fine Structure Constant, Electron Anomalous Magnetic Moment, and Quantum Electrodynamics

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based on the work carried out in collaboration with
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- The fine structure constant

$$\alpha = \frac{e^2}{2\epsilon_0 hc}$$

is a dimensionless fundamental constant of physics:

e = electric charge of electron,
ϵ_0 = dielectric constant of vacuum,
h = Planck constant,
c = velocity of light in vacuum.

- Since α is basically measure of magnitude of e, it can be measured by any physical system involving electron directly or indirectly.

- Some high precision measurements of α:

Mohr,Taylor,Newell, RMP 80, 633 (2008)

$$\alpha^{-1}(\textit{ac Josephson}) = 137.035\,987\,5\,(43) \qquad [31\ \textit{ppb}]$$

$$\alpha^{-1}(\textit{quantum Hall}) = 137.036\,003\,0\,(25) \qquad [18\ \textit{ppb}]$$

$$\alpha^{-1}(\textit{neutron wavelength}) = 137.036\,007\,7\,(28) \quad [21\ \textit{ppb}]$$

$$\alpha^{-1}(\textit{atom interferometry}) = 137.036\,000\,0\,(11) \quad [7.7\ \textit{ppb}]$$

$$\alpha^{-1}(\textit{optical lattice}) = 137.035\,998\,83\,(91) \qquad [6.7\ \textit{ppb}]$$

- However, by far the most accurate α comes from the measurement of electron anomalous magnetic moment a_e and the theoretical calculation in quantum electrodynamics (QED) and Standard Model (SM):

$$\alpha^{-1}(a_e) = 137.035\,999\,085\ (12)(37)(33) \quad [0.37\ ppb]$$

where 12,37,33 are the uncertainties of 8th-order term, estimated 10th-order term, and measurement of a_e.

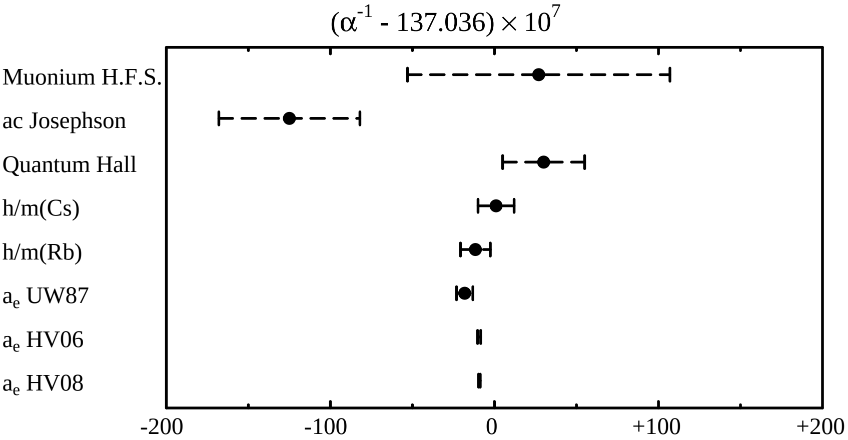


Figure: Comparison of various α^{-1} of high precision.

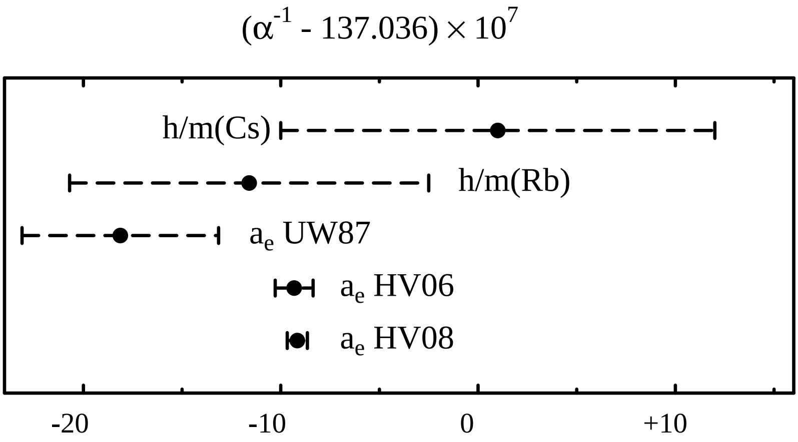


Figure: Magnification of the lower half of the last figure by factor 10.

- $\alpha(a_e[HV08])$ is 18 times more accurate than $\alpha(\mathrm{Rb})$.

Brief review of QED

- I will discuss how such precise α is obtained from a_e, and possible implication on quantum mechanics (including QED and SM).
- Of course quantum mechanics is the basis of our understanding of material world and the foundation of many modern technologies.

Heisenberg, Z. Physik 33, 879 (1925)
Schrödinger, Ann. der Phys. 81, 109 (1926)

- However, it was known from the beginning that it is applicable only to slowly moving particle, slow compared with the velocity of light c.
- One way to remove this restriction was found by Dirac who discovered an equation, Dirac equation, which works at any velocity.

Dirac, Proc. Roy. Soc. (London) A117, 610 (1928)

- Among many successes of Dirac equation is the prediction that gyromagnetic ratio g of electron is equal to 2, which agreed with measurement.
- However, Dirac eq. had puzzling feature that it appeared to predict presence of negative energy states.
- It turned out that these states can be interpreted as positive energy states of positron (which has same mass but opposite charge to electron).
- This interpretation was justified experimentally by the discovery of positron in cosmic ray experiment.

  Anderson, Phys. Rev. 41, 405 (1932).

- Formally this is equivalent to treating electron and positron as operators of quantized Dirac field.

- By 1929 relativistic quantum field theory (QED), which describes the interaction of electron field and electromagnetic field, was formulated.

Heisenberg, Pauli, Z. Physik, 56, 1 (1929)
Dirac, Proc. Roy. Soc. (London) A136, 453 (1932)

- Calculation in 2nd order perturbation of QED for processes such as Compton scattering (Klein-Nishina formula), Bremsstrahlung, and atomic energy levels, agreed with experiments within few percents.
- However, when one tried to improve theory by including higher order effects, one ran into strange situation that result becomes infinitely large.
- QED was thus regarded as seriously sick for many years.
- (In 1939 Kramers was telling people that the divergence problem might be solved by "renormalization". But he had no idea how to work it out.).

- The key to solution was provided in 1947 by two experiments which showed that predictions of Dirac eq. require tiny but non-vanishing corrections, thanks to much improved measurement precision due to advances in microwave technology.
- One is Lamb shift of hydrogen atom:
  $2S_{1/2}$ level is about 1050 MHz higher than $2P_{1/2}$ level, whereas Dirac eq. predicts that they are at the same level (i.e., degenerate).

  Lamb, Retherford, Phys. Rev. 72, 241 (1947)
- The other is Zeeman splitting of Ga atom which showed that g-factor of the electron is about 0.1 % larger than the prediction of Dirac eq., i.e., electron has anomalous magnetic moment:

$$a_e \equiv (g-2)/2 = 0.001\,15\,(4).$$

Kusch, Foley, PR 72, 1256 (1947)

- These experiments forced people to realize that, mass and charge parameters of QED must be reinterpreted as observed mass and charge minus their radiative corrections.
- When QED is thus renormalized, these divergences disappear from calculated physical processes.
- Bethe applied this idea within nonrelativistic framework and obtained the Lamb shift in rough agreement with measurement.

  Bethe, Phys. Rev. 72, 339 (1947).
- For unambiguous treatment of renormalization, however, it is necessary to have relativistic formulation.
- Note: Relation of energy and mass is not fixed without relativity.

- Such relativistic formulation was being developed by Tomonaga and Schwinger, unknown to each other.

Tomonaga, RIKEN-IHO 22, 545 (1943) in Japanese; Prog. Theor. Phys. 1, 27 (1946)
Koba, Tomonaga, Prog. Theor. Phys. 2, 218 (1947)
Schwinger, Phys. Rev. 73, 416 (1948); 74, 1439 (1948); 75, 651 (1949); 76, 790 (1949)

- Schwinger applied it to a_e and obtained

$$a_e = \frac{\alpha}{2\pi} = 0.001\ 161\ldots.$$

in excellent agreement with the measurement.

Schwinger, PR 73, 416L (1948); PR 75, 898 (1949)

- Together with Bethe's work on the Lamb shift, this provided firm experimental support for renormalized QED.

- Over 60 years since then, precision of $g-2$ measurement has been improved by eight orders of magnitude (spin precession, Penning trap).
- Theory has also been improved by similar order of magnitude.
- Latest innovations and improvements of the Penning trap method by Gabrielse's groupt at Harvard led to the precision of 0.24 parts per billion:

$$a_e(\text{exp}) = 1\,159\,652\,180.73\,(0.28) \times 10^{-12} \quad [0.24\,ppb]$$

Hanneke,Fogwell,Gabrielse, PRL 100, 120801 (2008)

- Their cylindrical Penning trap is shown next.

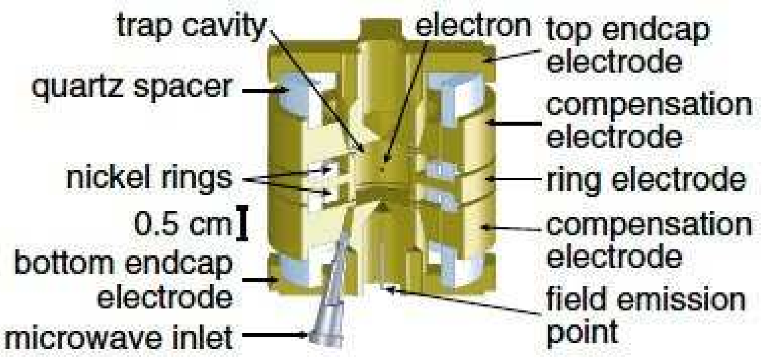
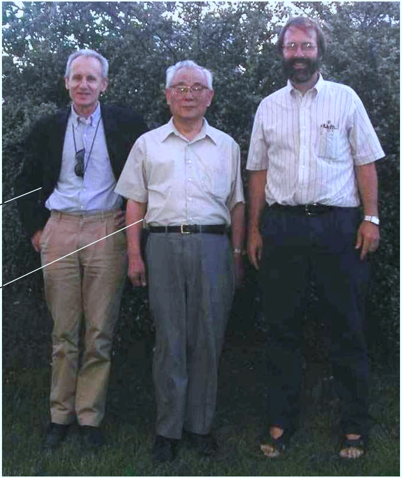


Figure: Cylindrical Penning Trap of Harvard experiment

At Gabrielse's apartment near CERN

Remiddi Kinoshita Gabrielse



Basking in the reflected glow of theorists

Feynman-Dyson method

- Simple theoretical structure and availability of precise measurement makes a_e particularly suitable for high precision test of validity of QED.
- However, computational methods of Tomonaga and Schwinger were too cumbersome to handle theory to comparable precision.
- It is diagrammatic method and use of Feynman propagator invented by Feynman and elaborated by Dyson that simplifies the calculation enormously and enables us to evaluate $g - 2$ systematically to high orders.

Feynman, Phys. Rev. 74, 1430, (1948); 76, 769 (1949)
Dyson, Phys. Rev. 75, 486, 1736 (1949)

(a) Feynman explaining parton theory, 1962

(b) Freeman Dyson

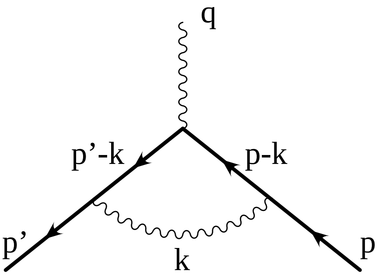


Figure: Feynman diagram: Radiative correction to the scattering of electron from momentum p to momentum $p^{'}$ by the potential (represented by q).

- Feynman-Dyson rule gives:

$$\lim_{\epsilon\to+0}\int d^4k\bar{u}(p^{'})\gamma^{\nu}\frac{i}{\not{p}'-\not{k}-m+i\epsilon}\gamma^{\mu}\frac{i}{\not{p}-\not{k}-m+i\epsilon}\gamma_{\nu}u(p)\frac{-i}{k^2+i\epsilon}$$

- Schwinger's formula will be $\sim$ 8 times longer.
- Tomonaga's formula will be even longer.

Following Dyson we can write a_e as
$a_e = a_e(\text{QED}) + a_e(\text{hadron}) + a_e(\text{electroweak})$, where

$$a_e(\text{QED}) = A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) + A_3(m_e/m_\mu, m_e/m_\tau)$$

$$A_i = A_i^{(2)}\left(\frac{\alpha}{\pi}\right) + A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + \ldots, i = 1, 2, 3$$

- First four A_1 terms are known analytically or by numerical integration

$A_1^{(2)} = 0.5$	1 Feynman diagram (analytic)
$A_1^{(4)} = -0.328\,478\,965\ \ldots$	7 Feynman diagrams (analytic)
$A_1^{(6)} = 1.181\,241\,456\ \ldots$	72 Feynman diagrams (analytic,numerical)

Laporta, Remiddi, PLB 379, 283 (1996)
Kinoshita, PRL 75, 4728 (1995)

$A_1^{(8)} = -1.914\,4\,(35)$	891 Feynman diagrams (numerical)

Kinoshita,Nio, PRD 73, 013003 (2006)
Aoyama,Hayakawa,Kinoshita,Nio, PRD 77, 053012 (2008)

- A_2 term is small but not negligible: $\sim 2.72 \times 10^{-12}$.
- A_3 term is completely negligible at present: ($\sim 2.4 \times 10^{-21}$).
- Hadronic and electroweak contributions (in SM) are also known
  a) $a_e(\text{hadron}) = 1.689\ (20)\ \times 10^{-12}$

  Jegerlehner, priv. com. 1996
  Krause, PLB 390, 392 (1997)
  Nyfeller, arXiv:0901.1172 [hep-ph]

  b) $a_e(\text{EW}) = 0.030\ \times 10^{-12}$

  Czarnecki *et al*., PRL 76, 3267 (1996)

- If one assumes $|A_1^{(10)}| < 4.6$

Mohr,Taylor,Newell, RMP 80, 633 (2008)

for the unknown 10th-order term, one obtains

$$a_e(\mathrm{Rb}) = 1\,159\,652\,182.79\,(0.11)(0.37)(7.72) \times 10^{-12},$$
$$a_e(\mathrm{exp}) - a_e(\mathrm{Rb}) = -2.06\,(7.72) \times 10^{-12}.$$

where

$$\alpha^{-1}(\mathrm{Rb}) = 137.035\,998\,84\,(91). \quad [6.7\ ppb],$$

is the value obtained by an optical lattice method.

P. Cladé et al., PRA 74, 052109 (2006)

- Uncertainty 0.11 of $A_1^{(8)}$ and guestimated error 0.37 of $A_1^{(10)}$ are much smaller than 7.72 of the best measured α available.
- Thus unknown $A_1^{(10)}$ does not appear to be a serious problem.

- To put it somewhat differently, non-QED α, even the best one, is too crude to test QED to the precision achieved by theory and measurement of a_e.
- Thus it makes more sense to test QED by an alternative approach:

  Get α from theory and measurement of a_e
  and compare it with other α's.

- This yields

$$\alpha^{-1}(a_e) = 137.035\,999\,085\,(12)(37)(33) \quad [0.37\ ppb],$$

  where 12, 37, 33 are uncertainties of 8th-order, 10th-order, and a_e(exp).
- Now the unknown $A_1^{(10)}$ becomes the largest source of uncertainty.

- Harvard group is working to reduce measurement error.
- Further progress of theory is not possible unless $A_1^{(10)}$ is actually calculated.
- Thus we began working on the 10th-order term more than 7 years ago.
  12672 Feynman diagrams contribute to $A_1^{(10)}$.
  9080 Feynman diagrams contributing to $A_2^{(10)}$.
- Clearly this is a gigantic project, requiring systematic and highly automated approach.
  First step: Classify them into gauge-invariant sets.
  There are 32 gauge-invariant sets within 6 supersets.

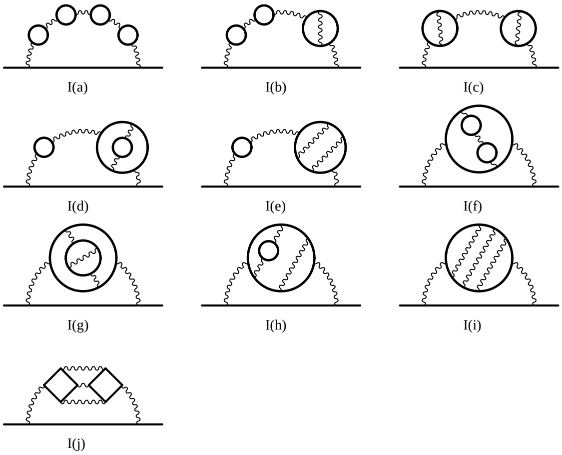


Figure: Diagrams of Superset I.

Set I consists of 10 subsets, all built from a second-order vertex. Solid lines represent electron propagating in magnetic field. Wavy lines represent photons. 208 diagrams contribute to $A_1^{(10)}$. 498 contribute to $A_2^{(10)}(m_e/m_\mu)$.

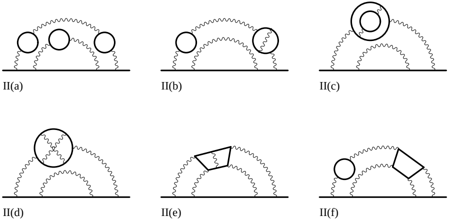


Figure: Diagrams of Superset II.

Set II is built from fourth-order proper vertices. 600 diagrams contribute to $A_1^{(10)}$. 1176 diagrams contribute to $A_2^{(10)}(m_e/m_\mu)$.

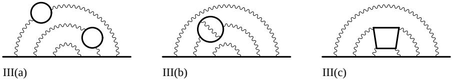


Figure: Diagrams of Superset III.

Set III is built from sixth-order proper vertices. 1140 diagrams contribute to $A_1^{(10)}$. 1740 diagrams contribute to $A_2^{(10)}(m_e/m_\mu)$.

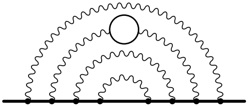


Figure: Diagrams of Superset IV.

Set IV is built from eighth-order proper vertices. 2072 diagrams contribute to both $A_1^{(10)}$ and $A_2^{(10)}(m_e/m_\mu)$.

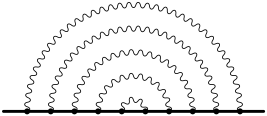


Figure: Diagrams of Superset V.

Set V consists of 10th-order proper vertices with no closed lepton loop. 6354 diagrams contribute to $A_1^{(10)}$. No contribution to $A_2^{(10)}(m_e/m_\mu)$.

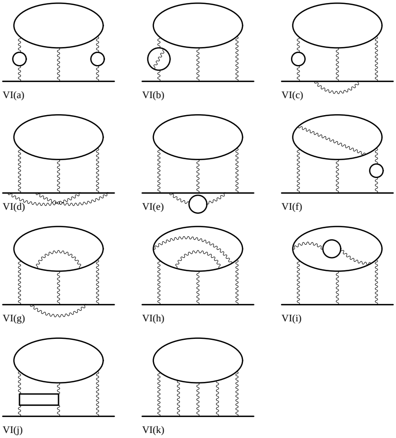


Figure: Diagrams of Superset VI.

This set has 11 subsets, all containing *light-by-light-scattering* subdiagrams. 2298 diagrams contribute to $A_1^{(10)}$. 3594 contribute to $A_2^{(10)}(m_e/m_\mu)$.

- Actually, sets I(a), I(b), I(c), II(a), II(b) are fairly simple and even analytical results are known for some of them.

  S. Laporta, PLB 328, 522 (1994)
- Analytic integration of other diagrams is far more difficult. (Recall: Even 8th-order is not yet done.)
- Numerical approach is the only viable option at present.
- This is relatively easy for g-i sets containing only v-p loops.
- We have only to modify known lower-order integrals slightly.

- Diagrams containing light-light-scattering subdiagram are much harder to evaluate.
- But we managed to integrate some of them by 2006.

Kinoshita,Nio, Phys. Rev. D 73, 013003 (2006)

- Now we have evaluated all other sets with the help of automatic code generators *gencodevpN* and *gencodeLLN*.

Aoyama,Hayakawa,Kinoshita,Nio, Phys. Rev. D 78, 113006 (2008)
Aoyama,Hayakawa,Kinoshita,Nio,Watanabe, Phys. Rev. D 78, 053005 (2008)
Aoyama,Asano,Hayakawa,Kinoshita,Nio,Watanabe, Phys. Rev. D 81, 053009 (2010)

- Tables on next pages summarize the current status, including those published already.

Table: Numerical values of diagrams of Set I. Preliminary results are indicated by red

Set	$A_1^{(10)}$	$A_2^{(10)}(m_e/m_\mu)$	$A_2^{(10)}(m_\mu/m_e)$	$A_2^{(10)}(m_\mu/m_\tau)$	$A_3^{(10)}(m_\mu/m_e, m_\mu/m_\tau)$
$I(a)$	0.000 470 94 (6)	0.000 000 28 (0)	22.567 05 (25)	0.000 038 (1)	0.015 200 (7)
$I(b)$	0.007 010 8 (7)	0.000 001 88 (0)	30.667 54 (33)	0.000 271 (1)	0.020 176 (8)
$I(c)$	0.023 468 (2)	0.000 002 67 (0)	5.141 38 (15)	0.003 936 (1)	0.002 331 (2)
$I(d)$	0.004 451 7 (5)	0.000 000 39 (0)	8.892 07 (102)	0.000 057 (1)	0.001 225 (8)
$I(e)$	0.010 296 (4)	0.000 001 60 (0)	−1.219 20 (71)	0.000 232 (1)	0.002 372 (2)
$I(f)$	0.008 445 9 (14)	0.000 047 54 (7)	3.685 10 (13)	0.001 231 (6)	0.019 730 (13)
$I(g)^*$	0.028 569 (6)	0.000 023 49 (2)	2.607 87 (38)	0.001 697 (3)	0.002 721 (5)
$I(h)^*$	0.001 696 (13)	−0.000 010 56 (14)	−0.568 61 (104)	0.000 160 (5)	0.001 978 (18)
$I(i)^*$	0.017 47 (11)	0.000 001 67 (3)	0.087 6 (65)	0.000 237 (2)	absent
$I(j)$	0.000 397 5 (18)	0.000 002 3 (1)	−1.263 72 (14)	0.000 149 (2)	0.000 110 (5)

* Generated automatically by *gencodeN* and *gencodevpN*.

Columns 2, 3, 4, 5, 6 list mass-independent terms of a_e, terms of a_e dependent on m_e/m_μ, electron loop contributions to a_μ, tau-lepton loop contribution to a_μ, and terms of a_μ dependent on both m_μ/m_e and m_μ/m_τ, respectively.

Table: Numerical values of diagrams of Sets II, III, and IV. Preliminary results are indicated by red.

Set	$A_1^{(10)}$	$A_2^{(10)}(m_e/m_\mu)$	$A_2^{(10)}(m_\mu/m_e)$	$A_2^{(10)}(m_\mu/m_\tau)$	$A_3^{(10)}(m_\mu/m_e, m_\mu/m_\tau)$
$II(a)$	0.004 13 (9)	−0.000 513 (8)	−70.471 7 (38)	−0.086 88 (74)	−0.523 9 (22)
$II(b)$	−0.054 22 (4)	−0.000 630 (7)	−34.771 5 (26)	0.004 71 (15)	0.035 90 (94)
$II(c)$*	−0.116 39 (13)	−0.000 366 (1)	−3.999 5 (72)	−0.125 110 (78)	−0.051 85 (32)
$II(d)$*	−0.243 00 (29)	−0.000 097 (1)	0.513 7 (85)	−0.007 69 (4)	absent
$II(e)$*	−1.344 9 (10)	−0.000 465 (4)	3.265 (12)	−0.038 1 (2)	absent
$II(f)$	−2.434 6 (16)	−0.005 94 (49)	−77.465 (12)	−0.267 5 (65)	−0.505 (27)
$III(a)$*	2.127 48 (22)	−0.013 69 (11)	109.022 7 (32)	1.021 07 (48)	absent
$III(b)$*	3.326 94 (33)	0.002 730 (35)	11.936 7 (46)	0.142 6 (12)	absent
$III(c)$	4.906 (22)	0.003 49 (91)	7.20 (24)	0.202 5 (37)	absent
IV*	−7.736 0 (52)	−0.009 77 (16)	−38.81 (17)	−0.441 3 (40)	absent

* Generated automatically by *gencodeN* and *gencodevpN*.

Columns 2, 3, 4, 5, 6 list mass-independent terms of a_e, terms of a_e dependent on m_e/m_μ, electron loop contributions to a_μ, tau-lepton loop contribution to a_μ, and terms of a_μ dependent on both m_μ/m_e and m_μ/m_τ, respectively.

Set VI containing light-by-light-scattering loop(s) are more difficult.

Table: Numerical values of diagrams of Set VI. Preliminary results are indicated by red.

Set	$A_1^{(10)}$	$A_2^{(10)}(m_e/m_\mu)$	$A_2^{(10)}(m_\mu/m_e)$	$A_2^{(10)}(m_\mu/m_\tau)$	$A_3^{(10)}(m_\mu/m_e, m_\mu/m_\tau)$
$VI(a)$	1.041 7 (4)	0.004 97 (29)	629.141 (12)	0.227 3 (48)	1.991 (71)
$VI(b)$	1.347 3 (3)	0.001 742 (47)	181.128 5 (51)	0.095 3 (11)	0.189 3 (32)
$VI(c)$	−2.592 2 (34)	−0.005 34 (47)	−36.576 (114)	−0.279 3 (77)	−0.478 7 (954)
$VI(d)$*	1.846 7 (70)	0.001 276 (76)	−7.983 (811)	0.081 77 (151)	absent
$VI(e)$	−0.431 2 (6)	−0.000 765 (40)	−4.322 (135)	−0.035 77 (57)	−0.118 2 (58)
$VI(f)$	0.770 3 (24)	−0.000 26 (26)	−38.16 (15)	0.119 (57)	0.173 (11)
$VI(g)$*	−1.590 4 (63)	−0.000 497 (29)	7.346 (489)	−0.044 51 (96)	absent
$VI(h)$*	0.179 2 (39)	0.000 045 (10)	−8.546 (231)	0.004 85 (46)	absent
$VI(i)$	−0.043 8 (11)	−0.000 508 (123)	−27.337 (115)	−0.004 53 (172)	−0.004 1 (77)
$VI(j)$	−0.228 8 (17)	−0.000 37 (35)	−25.505 (20)	−0.014 2 (63)	0.237 (14)
$VI(k)$	0.680 2 (38)	0.000 202 (24)	97.123 (62)	0.001 4 (17)	absent

* Generated automatically by *gencodeLLN*.

Columns 2, 3, 4, 5, 6 list mass-independent terms of a_e, terms of a_e dependent on m_e/m_μ, electron loop contributions to a_μ, tau-lepton loop contribution to a_μ, and terms of a_μ dependent on both m_μ/m_e and m_μ/m_τ, respectively.

Evaluation of Set V

- Largest and most difficult is Set V, which consists of 6354 Feynman diagrams, more than half of total: 12672.
- Luckily, Set V has nice feature that the sum $\Lambda^\nu(p,q)$ of 9 vertex diagrams, obtained by inserting an external vertex in the electron lines of a self-energy diagram $\Sigma(p)$, can be expressed in the form

$$\Lambda^\nu(p,q) \simeq -q_\mu [\frac{\partial \Lambda_\mu(p,q)}{\partial q_\nu}]_{q=0} - \frac{\partial \Sigma(p)}{\partial p_\nu}$$

  which is derived from the Ward-Takahashi identity.
- This enables us to cut no. of independent integrals to 706.
- Time-reversal symmetry reduces it to 389. They are shown in next page.

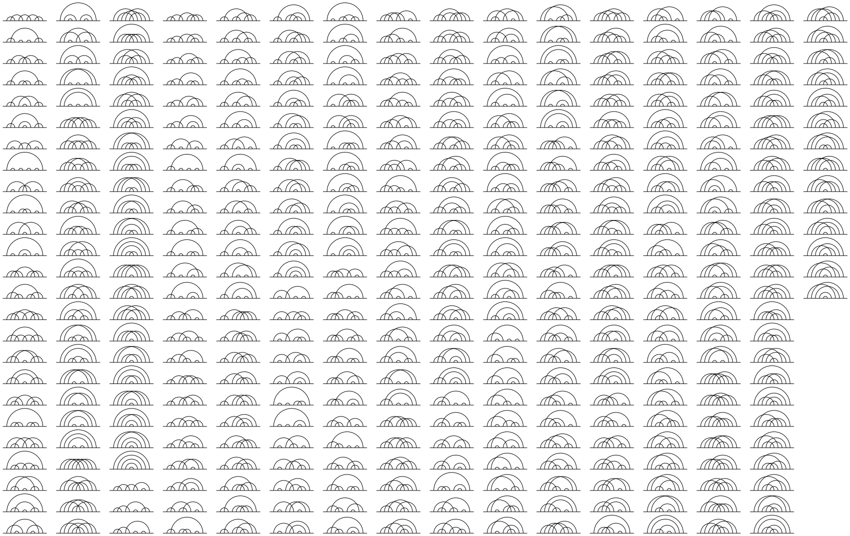
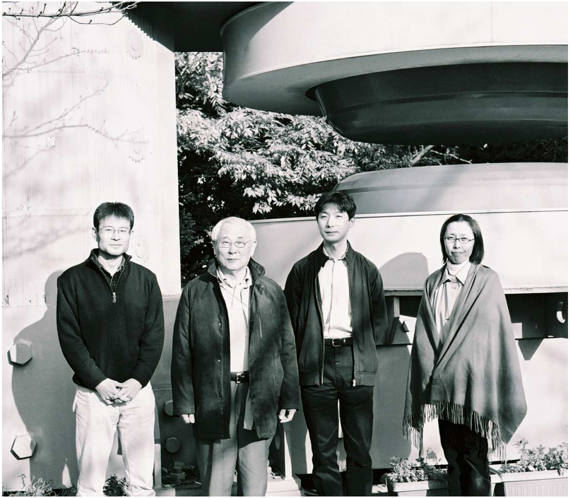


Figure: 389 self-energy diagrams representing 6354 vertex diagrams of Set V.

- It is to handle these diagrams that we developed the automatic code generator *gencodeN*.

T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Nucl. Phys. B **740**, 138 (2006); B **796**, 184 (2008).



gencodeN consists of several steps:

Step I: Diagram identification

- Specify diagram by vertex pairs connected by virtual photons. This information is stored as plain-text file Xabc (abc = 001, 002, ..., 389).
- Xabc defines not only the diagram itself but also identifies all UV- and IR-divergent subdiagrams.
- Implemented by Perl (and C++).

Step II: Construction of unrenormalized integrand

- Translate "Xabc" into mom. integral by Feynman-Dyson rule (by Perl). Output serves as input for Home-made analytic integration table written in FORM which turns it into a Feynman-parametric integral of the form

$$\int (dz) f(z), \quad (dz) \equiv \prod_{i=1}^{N} dz_i \delta(1 - \sum_{i=1}^{N} z_i).$$

- $f(z)$ is very complicated func. of z and seems nearly intractable.
- However, in terms of “building blocks” B_{ij}, A_i, U, V, it exhibits well-organized structure

$$f(z) = \frac{F_0(B_{ij}, A_i)}{U^2 V^{n-1}} + \frac{F_1(B_{ij}, A_i)}{U^3 V^{n-2}} + \cdots .$$

Step III: Construction of building blocks

- Express B_{ij}, U, V as polynomials of z_1, z_2, ..., z_N.
- U, B_{ij} are determined by network topology of loop momenta.
- They are obtained automatically by MAPLE and FORM:

$$\text{Xabc} \rightarrow B_{ij}, U, \ldots$$

- A_i is fraction of external momentum in line i, and satisfies Kirchhoff's loop law and junction law for "currents".
- Form of V is common to all diagrams of Set V:

$$V = \sum_{i}^{\text{electrons}} z_i(1 - A_i)m^2,$$

where m is lepton mass.

Step IV: Removal of UV divergence

- Renormalization must be performed exactly to 8th order.
- Our approach is subtractive renormalization.
- UV divergence arises from subdiagram S identified by $U \to 0$ for $\Sigma_S z_i \to 0$.
- UV-subtraction term is built from original integrand by procedure, called *K-operation*, which gives UV limits of B_{ij}, A_i, U, V based on a simple power-counting rule.
- Properties of terms generated by K-operation:
  - Pointwise subtraction of UV divergence.
  - Subtraction term factorizes analytically into product of lower-order quantities, a feature useful for cross-checking of different diagrams.
  - Gives only UV-divergent parts δm_n^{UV}, L_n^{UV}, B_n^{UV} of renorm. consts. δm_n, L_n, B_n.

Step V: Removal of IR divergence

- These integrals still suffer from IR divergence, logarithmic or worse.
- IR div. is characterized by $V \to 0$ in subdomain of integration.
- Linear (or worse) IR div., which is caused by the UV-finite term $\widetilde{\delta m_N} \equiv \delta m_N - \delta m_N^{UV}$ $(N > 2)$, is difficult to handle directly.
- This problem can be avoided by subtracting $\widetilde{\delta m_N}$ together with the UV-divergent δm_N^{UV} so that full mass renormalization is achieved in Step IV.

T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio,
Nucl. Phys. B **796**, 184 (2008).

- Remaining logarithmic IR divergence can be handled easily by *I-operation* defined by the IR power counting.

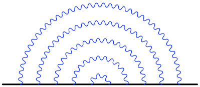
Example: Diagram X072

- An example: Diagram X072.
- Step I produces a file that contains just one-line statement

$$abcdeedcba$$

  which shows how photons a, b, c, d, e are attached to the electron line.
- This information is sufficient to generate complete instruction for Steps II and III for building 'unrenormalized' integral 'MX072' and also Steps IV and V for building UV-divergent and IR-divergent subintegrals.
- In our renormalization scheme all terms (represented symbolically in next page) are combined to form UV- and IR-finite piece "DMX072".

Divergence structure: an example



X072

```
DMX072 = MX072 - dmm47uv*M21s - Bm47uv*M2 - dm6buv*M4b2s - B6buv*M4b - dm4buv*M6b3s - B4buv*M6b - dm2uv*Mm474s - B2uv*Mm47 + dm6buv*
dm21suv*M21s + B6buv*dm2uv*M21s + B6buv*B2uv*M2 + dm4buv*dm4b2suv*M21s + B4buv*dm4buv*M21s + B4buv*B4buv*M2 + dm2uv*dm6b3suv*M21s +
B2uv*dm6buv*M21s + B2uv*B6buv*M2 + dm4buv*dm21suv*M4b2s + B4buv*dm2uv*M4b2s + B4buv*B2uv*M4b + dm2uv*dm4b2suv*M4b2s + B2uv*dm4buv*M4
b2s + B2uv*B4buv*M4b + dm2uv*dm21suv*M6b3s + B2uv*dm2uv*M6b3s + B2uv*B2uv*M6b - dm4buv*dm21suv*dm21suv*M21s - B4buv*dm2uv*dm21suv*M2
1s - B4buv*B2uv*dm2uv*M21s - B4buv*B2uv*B2uv*M2 - dm2uv*dm4b2suv*dm21suv*M21s - B2uv*dm4buv*dm21suv*M21s - B2uv*B4buv*dm2uv*M21s - B
2uv*B4buv*B2uv*M2 - dm2uv*dm21suv*dm4b2suv*M21s - B2uv*dm2uv*dm4b2suv*M21s - B2uv*B2uv*dm4buv*M21s - B2uv*B2uv*B4buv*M2 - dm2uv*dm21
suv*dm21suv*M4b2s - B2uv*dm2uv*dm21suv*M4b2s - B2uv*B2uv*dm2uv*M4b2s - B2uv*B2uv*B2uv*M4b + dm2uv*dm21suv*dm21suv*dm21suv*M21s + B2u
v*dm2uv*dm21suv*dm21suv*M21s + B2uv*B2uv*dm2uv*dm21suv*M21s + B2uv*B2uv*B2uv*dm2uv*M21s + B2uv*B2uv*B2uv*B2uv*M2 - dmm47ir*M21s + dm
6buv*dm21sir*M21s + B6buv*dm2ir*M21s + dm4buv*dm4b2sir*M21s + B4buv*dm4bir*M21s + dm2uv*dm6b3sir*M21s + B2uv*dm6bir*M21s - dm4buv*dm
21suv*dm21sir*M21s - B4buv*dm2uv*dm21sir*M21s - B4buv*B2uv*dm2ir*M21s - dm2uv*dm4b2suv*dm21sir*M21s - B2uv*dm4buv*dm21sir*M21s - B2u
v*B4buv*dm2ir*M21s - dm2uv*dm21suv*dm4b2sir*M21s - B2uv*dm2uv*dm4b2sir*M21s - B2uv*B2uv*dm4bir*M21s + dm2uv*dm21suv*dm21suv*dm21sir*
M21s + B2uv*dm2uv*dm21suv*dm21sir*M21s + B2uv*B2uv*dm2uv*dm21sir*M21s + B2uv*B2uv*B2uv*dm2ir*M21s - Mm47*L2vir + dm6buv*M21s*L2vir +
 B6buv*M2*L2vir + dm4buv*M4b2s*L2vir + B4buv*M4b*L2vir + dm2uv*M6b3s*L2vir + B2uv*M6b*L2vir - dm4buv*dm21suv*M21s*L2vir - B4buv*dm2u
v*M21s*L2vir - B4buv*B2uv*M2*L2vir - dm2uv*dm4b2suv*M21s*L2vir - B2uv*dm4buv*M21s*L2vir - B2uv*B4buv*M2*L2vir - dm2uv*dm21suv*M4b2s*
L2vir - B2uv*dm2uv*M4b2s*L2vir - B2uv*B2uv*M4b*L2vir + dm2uv*dm21suv*dm21suv*M21s*L2vir + B2uv*dm2uv*dm21suv*M21s*L2vir + B2uv*B2uv*
dm2uv*M21s*L2vir + B2uv*B2uv*B2uv*M2*L2vir - dm6bir*M4b2s + dm6bir*dm21suv*M21s + dm4buv*dm21sir*M4b2s + B4buv*dm2ir*M4b2s + dm2uv*d
m4b2sir*M4b2s + B2uv*dm4bir*M4b2s - dm4buv*dm21sir*dm21suv*M21s - B4buv*dm2ir*dm21suv*M21s - dm2uv*dm4b2sir*dm21suv*M21s - B2uv*dm4b
ir*dm21suv*M21s - dm2uv*dm21suv*dm21sir*M4b2s - B2uv*dm2uv*dm21sir*M4b2s - B2uv*B2uv*dm2ir*M4b2s + dm2uv*dm21suv*dm21sir*dm21suv*M21
s + B2uv*dm2uv*dm21sir*dm21suv*M21s + B2uv*B2uv*dm2ir*dm21suv*M21s - M6b*L4b2vir + M6b*L2vuv*L2vir + dm4buv*M21s*L4b2vir + B4buv*M2*
L4b2vir + dm2uv*M4b2s*L4b2vir + B2uv*M4b*L4b2vir - dm4buv*M21s*L2vuv*L2vir - B4buv*M2*L2vuv*L2vir - dm2uv*M4b2s*L2vuv*L2vir - B2uv*M
4b*L2vuv*L2vir - dm2uv*dm21suv*M21s*L4b2vir - B2uv*dm2uv*M21s*L4b2vir - B2uv*B2uv*M2*L4b2vir + dm2uv*dm21suv*M21s*L2vuv*L2vir + B2uv
*dm2uv*M21s*L2vuv*L2vir + B2uv*B2uv*M2*L2vuv*L2vir - dm4bir*M6b3s + dm4bir*dm4b2suv*M21s + dm4bir*dm21suv*M4b2s + dm2uv*dm21sir*M6b3
s + B2uv*dm2ir*M6b3s - dm4bir*dm21suv*dm21suv*M21s - dm2uv*dm21sir*dm4b2suv*M21s - B2uv*dm2ir*dm4b2suv*M21s - dm2uv*dm21sir*dm21suv*
M4b2s - B2uv*dm2ir*dm21suv*M4b2s + dm2uv*dm21sir*dm21suv*dm21suv*M21s + B2uv*dm2ir*dm21suv*dm21suv*M21s - M4b*L6b3vir + M4b*L4b2vuv*
L2vir + M4b*L2vuv*L4b2vir + dm2uv*M21s*L6b3vir + B2uv*M2*L6b3vir - M4b*L2vuv*L2vuv*L2vir - dm2uv*M21s*L4b2vuv*L2vir - B2uv*M2*L4b2vu
v*L2vir - dm2uv*M21s*L2vuv*L4b2vir - B2uv*M2*L2vuv*L4b2vir + dm2uv*M21s*L2vuv*L2vuv*L2vir + B2uv*M2*L2vuv*L2vuv*L2vir - M2*Lm474vir
+ M2*L6b3vuv*L2vir + M2*L4b2vuv*L4b2vir + M2*L2vuv*L6b3vir - M2*L4b2vuv*L2vuv*L2vir - M2*L2vuv*L4b2vuv*L2vir - M2*L2vuv*L2vuv*L4b2vi
r + M2*L2vuv*L2vuv*L2vuv*L2vir + dm6bir*dm21sir*M21s - dm4buv*dm21sir*dm21sir*M21s - B4buv*dm2ir*dm21sir*M21s - dm2uv*dm4b2sir*dm21s
ir*M21s - B2uv*dm4bir*dm21sir*M21s + dm2uv*dm21suv*dm21sir*dm21sir*M21s + B2uv*dm2uv*dm21sir*dm21sir*M21s + B2uv*B2uv*dm2ir*dm21sir*
M21s + dm6bir*M21s*L2vir - dm4buv*dm21sir*M21s*L2vir - B4buv*dm2ir*M21s*L2vir - dm2uv*dm4b2sir*M21s*L2vir - B2uv*dm4bir*M21s*L2vir +
 dm2uv*dm21suv*dm21sir*M21s*L2vir + B2uv*dm2uv*dm21sir*M21s*L2vir + B2uv*B2uv*dm2ir*M21s*L2vir + M6b*L2vir*L2vir - dm4buv*M21s*L2vir
*L2vir - B4buv*M2*L2vir*L2vir - dm2uv*M4b2s*L2vir*L2vir - B2uv*M4b*L2vir*L2vir + dm2uv*dm21suv*M21s*L2vir*L2vir + B2uv*dm2uv*M21s*L2
vir*L2vir + B2uv*B2uv*M2*L2vir*L2vir + dm4bir*dm4b2sir*M21s - dm4bir*dm21suv*dm21sir*M21s - dm2uv*dm21sir*dm4b2sir*M21s - B2uv*dm2ir
*dm4b2sir*M21s + dm2uv*dm21sir*dm21suv*dm21sir*M21s + B2uv*dm2ir*dm21suv*dm21sir*M21s + dm4bir*M4b2s*L2vir - dm4bir*dm21suv*M21s*L2v
ir - dm2uv*dm21sir*M4b2s*L2vir - B2uv*dm2ir*M4b2s*L2vir + dm2uv*dm21sir*dm21suv*M21s*L2vir + B2uv*dm2ir*dm21suv*M21s*L2vir + M4b*L4b
2vir*L2vir - M4b*L2vuv*L2vir*L2vir - dm2uv*M21s*L4b2vir*L2vir - B2uv*M2*L4b2vir*L2vir + dm2uv*M21s*L2vuv*L2vir*L2vir + B2uv*M2*L2vuv
*L2vir*L2vir + M2*L6b3vir*L2vir - M2*L4b2vuv*L2vir*L2vir - M2*L2vuv*L4b2vir*L2vir + M2*L2vuv*L2vuv*L2vir*L2vir + dm4bir*dm21sir*M4b2
s - dm4bir*dm21sir*dm21suv*M21s - dm2uv*dm21sir*dm21sir*M4b2s - B2uv*dm2ir*dm21sir*M4b2s + dm2uv*dm21sir*dm21sir*dm21suv*M21s + B2uv
*dm2ir*dm21sir*dm21suv*M21s + dm4bir*M21s*L4b2vir - dm4bir*M21s*L2vuv*L2vir - dm2uv*dm21sir*M21s*L4b2vir - B2uv*dm2ir*M21s*L4b2vir +
 dm2uv*dm21sir*M21s*L2vuv*L2vir + B2uv*dm2ir*M21s*L2vuv*L2vir + M4b*L2vir*L4b2vir - M4b*L2vir*L2vuv*L2vir - dm2uv*M21s*L2vir*L4b2vir
 - B2uv*M2*L2vir*L4b2vir + dm2uv*M21s*L2vir*L2vuv*L2vir + B2uv*M2*L2vir*L2vuv*L2vir + M2*L4b2vir*L4b2vir - M2*L4b2vir*L2vuv*L2vir -
M2*L2vuv*L2vir*L4b2vir + M2*L2vuv*L2vir*L2vuv*L2vir + M2*L2vir*L6b3vir - M2*L2vir*L4b2vuv*L2vir - M2*L2vir*L2vuv*L4b2vir + M2*L2vir*
L2vuv*L2vuv*L2vir - dm4bir*dm21sir*dm21sir*M21s + dm2uv*dm21sir*dm21sir*dm21sir*M21s + B2uv*dm2ir*dm21sir*dm21sir*M21s - dm4bir*dm21
sir*M21s*L2vir + dm2uv*dm21sir*dm21sir*M21s*L2vir + B2uv*dm2ir*dm21sir*M21s*L2vir - dm4bir*M21s*L2vir*L2vir + dm2uv*dm21sir*M21s*L2v
ir*L2vir + B2uv*dm2ir*M21s*L2vir*L2vir - M4b*L2vir*L2vir*L2vir + dm2uv*M21s*L2vir*L2vir*L2vir + B2uv*M2*L2vir*L2vir*L2vir - M2*L4b2v
ir*L2vir*L2vir + M2*L2vuv*L2vir*L2vir*L2vir - M2*L2vir*L4b2vir*L2vir + M2*L2vir*L2vuv*L2vir*L2vir - M2*L2vir*L2vir*L4b2vir + M2*L2vi
r*L2vir*L2vuv*L2vir + M2*L2vir*L2vir*L2vir*L2vir
```

UV- and IR-subtraction terms of X072.

A shell script controls flow of all Steps automatically:

(a) Get input information from data "Xabc" prepared in Step I.
(b) Build FORTRAN codes following Steps II, III, IV, V.
(c) Assemble FORTRAN codes, ready them for numerical integration.

- Numerical integration is carried out by adaptive-iterative Monte-Carlo routine VEGAS.

Lepage, J. Comput. Phys. 27, 192 (1978)

- Snapshot (11/02/2010) of calculation is shown in next pages in 12 columns, 4 columns for each diagram.
- Second column lists number of subtraction terms.
- Fourth column gives CPU time in minutes for $10^7 \times 50$ with 32 cores, in real*8, for mutual comparison. Many results listed are obtained with much higher statistics.

Table: Set V diagrams from X001 - X099

dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time
X 1	47	-0.2981(0.0327)	231	**X 2**	47	-5.9781(0.0462)	177	**X 3**	19	-0.1142(0.0094)	187
X 4	71	5.1280(0.0541)	241	**X 5**	43	1.1401(0.0377)	193	**X 6**	59	-5.2927(0.0433)	166
X 7	47	-3.4833(0.0467)	188	**X 8**	47	-16.5212(0.0554)	110	**X 9**	19	-2.8715(0.0584)	161
X 10	83	11.2762(0.0484)	105	**X 11**	43	6.0549(0.0453)	235	**X 12**	67	-9.3202(0.0304)	112
X 13	7	-1.3540(0.0038)	160	**X 14**	31	0.7833(0.0141)	228	**X 15**	2	2.1020(0.0019)	151
X 16	2	-0.9609(0.0019)	149	**X 17**	6	0.5174(0.0062)	150	**X 18**	6	0.0579(0.0069)	145
X 19	31	1.2183(0.0140)	192	**X 20**	134	-8.1361(0.0564)	90	**X 21**	11	-0.2967(0.0049)	162
X 22	79	0.9382(0.0433)	117	**X 23**	27	0.6047(0.0418)	220	**X 24**	75	-6.1010(0.0426)	119
X 25	39	-0.7824(0.0411)	188	**X 26**	95	-7.8186(0.0336)	98	**X 27**	15	-2.3190(0.0315)	166
X 28	71	4.5631(0.0588)	119	**X 29**	35	6.8839(0.0333)	167	**X 30**	67	-12.6108(0.0386)	99
X 31	2	2.2932(0.0029)	143	**X 32**	2	-0.2427(0.0013)	139	**X 33**	2	-1.3771(0.0014)	131
X 34	2	1.2539(0.0021)	140	**X 35**	2	-0.5838(0.0014)	145	**X 36**	11	0.2473(0.0064)	68
X 37	2	-0.7417(0.0020)	148	**X 38**	11	-0.2811(0.0049)	65	**X 39**	11	0.3164(0.0044)	208
X 40	47	1.4835(0.0314)	87	**X 41**	63	3.1418(0.0577)	175	**X 42**	119	-4.1234(0.0418)	84
X 43	15	-2.8829(0.0356)	149	**X 44**	59	4.4462(0.0399)	105	**X 45**	43	3.4311(0.0324)	118
X 46	95	-7.7361(0.0446)	88	**X 47**	2	-4.4551(0.0033)	125	**X 48**	2	-0.8051(0.0016)	135
X 49	2	-0.0295(0.0013)	130	**X 50**	2	-1.2222(0.0018)	123	**X 51**	2	-0.1733(0.0020)	148
X 52	11	0.9875(0.0094)	68	**X 53**	2	0.3646(0.0015)	144	**X 54**	11	-0.4924(0.0070)	65
X 55	2	-0.3634(0.0014)	146	**X 56**	11	-0.2408(0.0054)	68	**X 57**	23	2.6504(0.0164)	113
X 58	44	-5.1538(0.0331)	57	**X 59**	23	2.1860(0.0176)	142	**X 60**	92	-3.2758(0.0480)	90
X 61	68	-3.7959(0.0325)	100	**X 62**	161	5.9124(0.0428)	83	**X 63**	6	3.3563(0.0086)	143
X 64	6	-0.2763(0.0069)	145	**X 65**	6	0.1748(0.0055)	156	**X 66**	26	-3.5299(0.0396)	87
X 67	50	-1.7091(0.0660)	130	**X 68**	98	2.7344(0.0491)	86	**X 69**	18	-1.1586(0.0260)	110
X 70	70	3.2263(0.0329)	88	**X 71**	54	3.6918(0.0215)	100	**X 72**	134	-5.5392(0.0455)	86
X 73	47	3.4045(0.0448)	190	**X 74**	47	4.3918(0.0477)	199	**X 75**	47	-8.1355(0.0496)	165
X 76	19	-5.2424(0.0230)	176	**X 77**	39	3.2616(0.0443)	226	**X 78**	39	0.9403(0.0453)	225
X 79	71	5.4206(0.0466)	205	**X 80**	43	0.5166(0.0502)	206	**X 81**	59	-5.6569(0.0470)	152
X 82	47	-8.5074(0.0637)	202	**X 83**	47	18.7382(0.0475)	161	**X 84**	19	8.9855(0.0278)	173
X 85	39	-2.2692(0.0447)	217	**X 86**	39	0.5038(0.0442)	182	**X 87**	77	-16.5708(0.0650)	134
X 88	43	-5.2642(0.0480)	204	**X 89**	63	12.6876(0.0446)	138	**X 90**	19	1.5108(0.0294)	160
X 91	39	-1.8168(0.0486)	235	**X 92**	39	2.1022(0.0454)	194	**X 93**	7	-1.7604(0.0050)	160
X 94	15	-1.0460(0.0099)	194	**X 95**	7	0.5791(0.0043)	159	**X 96**	31	1.2849(0.0179)	196
X 97	17	4.7894(0.0593)	168	**X 98**	33	-1.9365(0.0370)	169	**X 99**	39	3.0813(0.0434)	195

Table: Set V diagrams from X100 - X198

dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time
X100	77	-15.3143(0.0627)	136	**X101**	15	-0.2625(0.0093)	188	**X102**	31	-1.3912(0.0312)	235
X103	31	0.8229(0.0193)	216	**X104**	79	6.4712(0.0581)	165	**X105**	35	3.0633(0.0487)	191
X106	71	-11.5433(0.0492)	136	**X107**	43	-4.6615(0.0584)	200	**X108**	63	12.9649(0.0440)	142
X109	17	0.0220(0.0440)	166	**X110**	35	1.9409(0.0417)	183	**X111**	33	3.4365(0.0688)	175
X112	71	-11.8915(0.0481)	136	**X113**	39	-4.4510(0.0545)	162	**X114**	63	11.0810(0.0468)	122
X115	7	-0.5947(0.0065)	158	**X116**	7	1.8059(0.0050)	161	**X117**	7	0.3232(0.0045)	157
X118	15	-3.2225(0.0106)	171	**X119**	15	-0.1055(0.0113)	197	**X120**	31	1.7913(0.0158)	194
X121	7	-0.8630(0.0044)	167	**X122**	7	-0.7414(0.0042)	162	**X123**	15	-3.3339(0.0075)	176
X124	29	11.1936(0.0631)	107	**X125**	31	0.7481(0.0189)	221	**X126**	59	-1.2410(0.0625)	139
X127	15	1.1349(0.0059)	192	**X128**	31	0.5916(0.0129)	198	**X129**	31	1.4312(0.0123)	220
X130	59	-1.5371(0.0393)	138	**X131**	59	3.1603(0.0727)	191	**X132**	101	-8.8220(0.0588)	125
X133	17	2.6477(0.0423)	172	**X134**	33	-0.4814(0.0641)	170	**X135**	33	1.0868(0.0659)	180
X136	65	-7.5387(0.0569)	137	**X137**	45	-2.5009(0.0739)	131	**X138**	85	10.1410(0.0480)	105
X139	47	14.8592(0.0502)	205	**X140**	39	-2.7411(0.0465)	156	**X141**	74	-12.5628(0.0680)	165
X142	43	-1.6913(0.0700)	189	**X143**	61	10.3414(0.0433)	154	**X144**	83	23.7308(0.0662)	103
X145	67	-18.6357(0.0471)	111	**X146**	39	-2.3801(0.0690)	200	**X147**	15	1.1276(0.0223)	149
X148	31	-1.3144(0.0364)	168	**X149**	17	-8.3912(0.0309)	164	**X150**	33	2.8037(0.0614)	195
X151	87	-10.8607(0.0603)	94	**X152**	77	14.6570(0.0480)	107	**X153**	77	14.9037(0.0559)	106
X154	67	-20.5911(0.0539)	102	**X155**	15	4.9510(0.0225)	169	**X156**	31	-0.7363(0.0454)	172
X157	32	-11.8522(0.0408)	69	**X158**	65	0.4466(0.0539)	83	**X159**	65	0.2208(0.0733)	84
X160	116	14.0278(0.0599)	96	**X161**	71	7.7606(0.0428)	102	**X162**	95	-12.8160(0.0408)	95
X163	19	6.6451(0.0702)	173	**X164**	19	-12.0134(0.0639)	137	**X165**	15	-2.1380(0.0114)	181
X166	15	-2.2856(0.0121)	169	**X167**	29	12.1602(0.0338)	115	**X168**	17	3.3651(0.0573)	158
X169	25	-6.9274(0.0247)	106	**X170**	39	0.2847(0.0573)	178	**X171**	39	-2.6059(0.0529)	164
X172	31	1.4301(0.0225)	214	**X173**	59	0.2731(0.0737)	237	**X174**	35	1.8660(0.0602)	177
X175	51	-1.8412(0.0397)	151	**X176**	7	0.7651(0.0184)	153	**X177**	15	-0.0111(0.0352)	149
X178	5	0.7079(0.0038)	153	**X179**	2	-0.4378(0.0034)	142	**X180**	11	0.0242(0.0044)	147
X181	6	-4.3571(0.0146)	142	**X182**	12	1.2875(0.0157)	167	**X183**	7	-0.0179(0.0186)	140
X184	31	0.4381(0.0637)	170	**X185**	5	-0.1313(0.0050)	142	**X186**	23	1.1634(0.0049)	195
X187	6	1.2832(0.0128)	137	**X188**	24	1.8185(0.0232)	190	**X189**	17	-3.7335(0.0226)	162
X190	33	-2.4993(0.0359)	193	**X191**	13	0.1938(0.0246)	150	**X192**	25	2.4665(0.0438)	175
X193	15	-4.2494(0.0175)	139	**X194**	27	-0.8543(0.0652)	163	**X195**	2	-1.0665(0.0045)	158
X196	2	-2.0375(0.0029)	130	**X197**	2	-0.3870(0.0022)	148	**X198**	5	-2.3452(0.0027)	132

Table: Set V diagrams from X199 - X297

dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time
X199	5	1.0493(0.0038)	145	**X200**	11	0.0092(0.0043)	151	**X201**	2	-0.4877(0.0037)	140
X202	2	1.9243(0.0030)	128	**X203**	2	0.9037(0.0023)	147	**X204**	11	-1.9324(0.0038)	158
X205	5	-0.9038(0.0049)	152	**X206**	23	1.6447(0.0065)	191	**X207**	5	0.2894(0.0042)	163
X208	11	0.5215(0.0040)	154	**X209**	5	0.1444(0.0040)	160	**X210**	23	0.7653(0.0049)	190
X211	23	5.1027(0.0348)	119	**X212**	41	-0.3297(0.0554)	173	**X213**	6	-2.4132(0.0118)	145
X214	12	0.6646(0.0142)	140	**X215**	6	0.1151(0.0120)	145	**X216**	24	-1.1993(0.0239)	164
X217	18	-2.2056(0.0537)	106	**X218**	30	-1.7370(0.0670)	130	**X219**	39	1.3630(0.0488)	173
X220	59	-2.4828(0.0511)	210	**X221**	35	0.6897(0.0347)	167	**X222**	51	0.8242(0.0386)	143
X223	116	17.4832(0.0543)	100	**X224**	31	2.4650(0.0232)	202	**X225**	23	0.2928(0.0098)	216
X226	13	1.0518(0.0231)	153	**X227**	25	0.6828(0.0398)	181	**X228**	75	-6.7788(0.0664)	108
X229	35	-1.9956(0.0651)	204	**X230**	71	15.6775(0.0549)	109	**X231**	11	-0.7467(0.0058)	177
X232	23	0.4010(0.0116)	215	**X233**	31	8.5433(0.0434)	79	**X234**	63	-2.4968(0.0460)	97
X235	23	0.7040(0.0100)	245	**X236**	63	2.0658(0.0381)	112	**X237**	113	-13.0105(0.0645)	104
X238	25	1.4003(0.0391)	180	**X239**	69	-2.8983(0.0645)	108	**X240**	93	10.9598(0.0571)	96
X241	43	13.8540(0.0634)	210	**X242**	68	-10.5143(0.0659)	195	**X243**	57	3.8884(0.0582)	176
X244	35	-3.3003(0.0654)	200	**X245**	27	0.0824(0.0338)	176	**X246**	29	-0.4379(0.0365)	192
X247	39	15.9448(0.0552)	213	**X248**	31	-1.9496(0.0459)	161	**X249**	13	3.9940(0.0159)	156
X250	27	-0.8949(0.0402)	172	**X251**	27	-1.2982(0.0300)	170	**X252**	56	-10.9812(0.0691)	141
X253	113	17.8089(0.0672)	86	**X254**	29	2.1746(0.0392)	192	**X255**	43	8.1509(0.0541)	140
X256	93	-14.0506(0.0543)	93	**X257**	7	5.6299(0.0259)	133	**X258**	7	-0.4470(0.0168)	155
X259	5	0.0160(0.0049)	141	**X260**	5	-0.4007(0.0036)	165	**X261**	6	6.3373(0.0172)	132
X262	6	-2.2800(0.0140)	153	**X263**	7	-2.7605(0.0144)	142	**X264**	15	4.7945(0.0346)	156
X265	5	-0.6741(0.0034)	141	**X266**	11	0.1179(0.0048)	167	**X267**	6	-0.6336(0.0099)	138
X268	12	0.1262(0.0191)	156	**X269**	15	-0.6542(0.0308)	171	**X270**	31	-1.5919(0.0607)	173
X271	11	0.2415(0.0053)	205	**X272**	23	-0.7339(0.0093)	217	**X273**	13	-2.0001(0.0240)	165
X274	25	0.8899(0.0406)	175	**X275**	2	-0.7434(0.0044)	126	**X276**	2	-0.5544(0.0028)	133
X277	2	2.7843(0.0015)	185	**X278**	5	-0.1559(0.0044)	144	**X279**	5	0.8231(0.0038)	164
X280	2	-1.0096(0.0046)	130	**X281**	5	-1.3724(0.0041)	154	**X282**	5	0.4841(0.0034)	148
X283	11	-0.0505(0.0042)	167	**X284**	2	-0.2711(0.0032)	159	**X285**	5	0.0169(0.0039)	152
X286	11	0.7775(0.0038)	186	**X287**	23	0.1874(0.0068)	190	**X288**	6	4.1604(0.0152)	130
X289	6	-1.5135(0.0130)	152	**X290**	6	-3.7248(0.0117)	143	**X291**	12	1.5878(0.0178)	158
X292	12	0.9126(0.0149)	163	**X293**	24	-1.1657(0.0266)	167	**X294**	7	-3.3322(0.0166)	150
X295	7	1.7876(0.0186)	151	**X296**	5	0.5448(0.0046)	170	**X297**	5	-0.4792(0.0047)	160

Table: Set V diagrams from X298 - X389

dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time	dgrm	sub.	Value (Error)	time
X298	6	-1.8909(0.0115)	158	**X299**	6	-0.2647(0.0122)	155	**X300**	29	-9.3516(0.0580)	84
X301	31	-1.0812(0.0672)	184	**X302**	59	-1.8824(0.0523)	105	**X303**	2	0.3213(0.0025)	152
X304	5	-0.3422(0.0049)	165	**X305**	5	0.4619(0.0040)	167	**X306**	23	0.1582(0.0226)	71
X307	47	-0.1151(0.0397)	88	**X308**	6	1.8367(0.0145)	154	**X309**	26	-4.2650(0.0376)	84
X310	50	-0.0629(0.0737)	100	**X311**	15	-0.4378(0.0278)	165	**X312**	31	-1.1090(0.0534)	177
X313	11	0.9513(0.0043)	192	**X314**	23	0.7992(0.0070)	203	**X315**	13	-1.2886(0.0216)	163
X316	25	0.1050(0.0338)	187	**X317**	59	1.3935(0.0522)	106	**X318**	62	-8.7913(0.0674)	170
X319	47	0.7468(0.0678)	204	**X320**	11	0.5585(0.0045)	168	**X321**	23	-0.9154(0.0078)	237
X322	23	0.9205(0.0032)	189	**X323**	25	0.0954(0.0331)	173	**X324**	53	-8.8189(0.0510)	133
X325	107	11.6018(0.0571)	93	**X326**	17	-8.8868(0.0557)	170	**X327**	33	1.4993(0.0617)	192
X328	13	-0.2799(0.0191)	163	**X329**	25	-0.8929(0.0252)	180	**X330**	15	-4.8847(0.0546)	174
X331	27	4.4591(0.0628)	194	**X332**	33	2.8378(0.0722)	195	**X333**	65	6.7152(0.0634)	122
X334	47	5.2084(0.0554)	163	**X335**	37	-2.0700(0.0568)	139	**X336**	6	-0.7509(0.0076)	153
X337	12	-1.1895(0.0143)	146	**X338**	13	-1.8395(0.0208)	168	**X339**	25	0.4930(0.0283)	169
X340	53	-2.2777(0.0676)	93	**X341**	24	1.8004(0.0137)	156	**X342**	27	2.5993(0.0173)	157
X343	2	3.8805(0.0029)	149	**X344**	2	3.4147(0.0037)	122	**X345**	2	-1.0015(0.0024)	153
X346	2	0.2844(0.0037)	138	**X347**	2	-2.6792(0.0028)	149	**X348**	2	-0.4859(0.0038)	149
X349	5	2.0816(0.0043)	151	**X350**	2	1.4548(0.0023)	143	**X351**	5	0.2449(0.0034)	155
X352	2	-0.1319(0.0025)	157	**X353**	5	0.1884(0.0025)	169	**X354**	5	-2.0375(0.0025)	148
X355	11	-1.0637(0.0031)	162	**X356**	5	2.0708(0.0049)	161	**X357**	5	0.3634(0.0037)	168
X358	5	0.0333(0.0043)	165	**X359**	11	-0.1515(0.0046)	170	**X360**	11	-0.4709(0.0042)	182
X361	23	2.5319(0.0064)	201	**X362**	2	-0.5660(0.0036)	147	**X363**	2	-2.3416(0.0022)	141
X364	2	2.3899(0.0021)	147	**X365**	11	0.4884(0.0115)	62	**X366**	23	5.6077(0.0222)	65
X367	5	-0.7180(0.0049)	171	**X368**	23	-0.2878(0.0180)	68	**X369**	47	-3.2062(0.0395)	77
X370	5	-1.4791(0.0045)	155	**X371**	5	-0.0074(0.0042)	154	**X372**	11	-1.2875(0.0025)	162
X373	23	0.5684(0.0039)	183	**X374**	47	0.9446(0.0533)	147	**X375**	89	0.8509(0.0589)	68
X376	5	1.0369(0.0034)	164	**X377**	11	0.4192(0.0036)	188	**X378**	11	1.3081(0.0034)	176
X379	23	-0.3402(0.0052)	198	**X380**	47	-0.9354(0.0359)	76	**X381**	23	1.0677(0.0038)	187
X382	41	-1.6457(0.0389)	183	**X383**	6	-4.7039(0.0136)	151	**X384**	12	1.9230(0.0184)	158
X385	12	-0.6982(0.0141)	157	**X386**	24	0.7383(0.0244)	163	**X387**	50	1.7962(0.0748)	89
X388	24	-0.3893(0.0199)	163	**X389**	30	-0.2604(0.0589)	127				

Statistics of running Set V:

- 10–20 minutes for generating FORTRAN code for each diagram on HP Alpha.
- Typical integrand consists of 90,000 lines of FORTRAN code occupying more than 6 Megabytes.
- Evaluation of integral in real*8 with 10^7 sampling points $\times$ 50 iterations takes $2-4$ hours on 32 cores of RICC (RIKEN Integrated Clusters of Clusters).
- Evaluation in real*16 is about 10 times slower.
- Large runs: real*16, $10^9 \times 100$, takes about 24 days on 256 cores of RICC.

VI: Residual renormalization

- Integrals in these Tables are UV- and IR-finite, but not standard renormalized amplitudes.
- Thus finite adjustment, called residual renormalization, must be carried out to get observable $g-2$.
- Residual renormalization of all diagrams of Set V requires systematic handling of more than 10,000 integrals.

- Fortunately, they can be organized into 16 terms whose structures are readily recognizable in terms of lower-order quantities

$$
\begin{aligned}
a_{10} = \; & \Delta M_{10} \\
& + \Delta M_8 \left(-7\Delta LB_2\right) \\
& + \Delta M_6 \left(-5\,\Delta LB_4 + 20\,(\Delta LB_2)^2\right) \\
& + \Delta M_4 \left(-3\,\Delta LB_6 + 24\,\Delta LB_2\,\Delta LB_4 - 28\,(\Delta LB_2)^3 + 2\,\Delta L_{2\star}\,\Delta\delta m_4\right) \\
& + \Delta M_2 \Big(-\Delta LB_8 + 4\,(\Delta LB_4)^2 + 8\,\Delta LB_2\,\Delta LB_6 - 28\,(\Delta LB_2)^2\,\Delta LB_4 + 14\,(\Delta LB_2)^4 \\
& \qquad +2\,\Delta L_{2\star}\,\Delta\delta m_6 - 2\,\Delta L_{2\star}\,\Delta\delta m_{2\star}\,\Delta\delta m_4 - 16\,\Delta L_{2\star}\,\Delta LB_2\,\Delta\delta m_4 + \Delta L_{4\star}\,\Delta\delta m_4\Big)
\end{aligned}
$$

where ΔM_n is the finite part of the n-th order magnetic moment,
ΔLB_n is the sum of finite parts of the n-th order vertex renormalization constant ΔL_n and the wavefunction renormalization constant ΔB_n,
$\Delta\delta m_n$ is the finite part of the n-th order selfmass of the electron,
$\Delta L_{n\star}$ is obtained from ΔL_n by insertion of 2-vertex in the electron line.

- When residual renormalization is included entire FORTRAN codes becomes analytically exact. No approximation involved.
- Uncertainty of numerical value arises only from numerical integration, which is performed by adaptive-iterative Monte-Carlo routine VEGAS.

- Latest value (11/02/2010) of the sum of 389 integrals and residual renormalization terms is

$$A_1^{(10)}\textit{[Set V]} = 9.752\ (733) \quad \textit{[Preliminary]}$$

- Uncertainty is being reduced further.
- To obtain $A_1^{(10)}[all]$ we must add values of other 31 sets.
This leads to

$$A_1^{(10)}\textit{[all]} = 4.364\ (733) \quad \textit{[Preliminary]}$$

- This is still very crude but is already about 6 times more precise than the previous guestimate $|A_1^{(10)}| < 4.6$.

Improving $A_1^{(8)}$

- We are also working to reduce the uncertainty of $A_1^{(8)}$.
- The latest value (11/02/2010) is

$$A_1^{(8)} = -1.910\,8\;(25) \quad \textit{[Preliminary]}$$

- In terms of these values of $A_1^{(8)}$ and $A_1^{(10)}$ we find

$$\alpha^{-1}(a_e) = 137.035\,999\,132\;(9)(6)(33)$$
$$\textit{[0.254 ppb]}, \textit{[Preliminary]}$$

  where 9, 6, 33 are uncertainties of 8th-order, 10th-order, and a_e(exp).
- This is about 30 times more precise than $\alpha^{-1}(Rb)$.
- Further progress now depends on improving a_e(exp).

- Discoverers of QED regarded the renormalization procedure as a jelly-built temporary fix to be replaced by something better.

Tomonaga: Private communication.
Letter of Dyson to Gabrielse quoted in Physics Today (August 2006), p.15.

Gabrielse

From Freeman Dyson – One Inventor of QED

Dear Jerry,

... I love your way of doing experiments, and I am happy to congratulate you for this latest triumph. Thank you for sending the two papers.

Your statement, that QED is tested far more stringently than its inventors could ever have envisioned, is correct. As one of the inventors, I remember that we thought of QED in 1949 as a temporary and jerry-built structure, with mathematical inconsistencies and renormalized infinities swept under the rug. We did not expect it to last more than ten years before some more solidly built theory would replace it. We expected and hoped that some new experiments would reveal discrepancies that would point the way to a better theory. And now, 57 years have gone by and that ramshackle structure still stands. The theorists … have kept pace with your experiments, pushing their calculations to higher accuracy than we ever imagined. And you still did not find the discrepancy that we hoped for. To me it remains perpetually amazing that Nature dances to the tune that we scribbled so carelessly 57 years ago. And it is amazing that you can measure her dance to one part per trillion and find her still following our beat.

With congratulations and good wishes for more such beautiful experiments, yours ever, Freeman.

- In fact it was soon found that QED must be enlarged to include hadronic and weak interactions, which led to the Standard Model (SM).
- But jelly-built structure itself remained as the basic framework of SM.
- SM itself is generally regarded as temporary measure which requires further modification to accommodate new physics.
- Such modification is most likely to come from experiments at high energy accelerators such as LHC.
- However, impact of new physics on a_e may not necessarily be straightforward.

- As a matter of fact, it might have no detectable effect on $g-2$.
- Reason: Independent α's measured by non-QED means must also include the effect of new physics.
- Recall that masses and charges involved in ordinary QM cannot be correctly identified as physical mass and charge to the precision that requires radiative corrections.
- For proper interpretation the ordinary QM, by which these measurements of α are interpreted, must be extended to include these effects.
- Such a formulation was obtained for one- and two-body systems as the nonrelativistic limit of explicitly renormalized QED (or SM), unfortunately misnamed NRQED.

  Caswell, Lepage, PL 167B, 437 (1986)
- As far as I know, no attempt has been made thus far to extend it to many-particle systems.

- To conclude, the jelly-built structure still looks good at the precision exceeding 1 part in 10^9.
- If disagreement is detected at the next level of precision it might indicate that breakdown of SM comes, not necessarily from high energy region, but from an entirely unexpected direction.
- Unfortunately such an event may not be detectable until α is measured by some independent method with precision comparable to that of $\alpha(a_e)$.
- Until then, $\alpha(a_e)$ serves as the yardstick by which validity of other types of measurements and their theories is examined.